Ex13. 设A, B 都是$m \times n$ 阶矩阵且$rank(A) = rank(B)$, 证明: A 与B 等价.

证明. 设$rank(A) = rank(B) = r$, 则存在初等矩阵$P_1, P_2, \cdots, P_s$ 和$Q_1, Q_2, \cdots, Q_t$ $(s, t \geq 0)$ 使得

$$P_1 P_2 \cdots P_s A Q_1 Q_2 \cdots Q_t = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}.$$

存在初等矩阵$\tilde{P}_1, \tilde{P}_2, \cdots \tilde{P}_{\tilde{s}}$ 和$\tilde{Q}_1, \tilde{Q}_2, \cdots \tilde{Q}_{\tilde{t}}$ $(\tilde{s}, \tilde{t} \geq 0)$ 使得

$$\tilde{P}_1 \tilde{P}_2 \cdots \tilde{P}_{\tilde{s}} B \tilde{Q}_1 \tilde{Q}_2 \cdots \tilde{Q}_{\tilde{t}} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}.$$

所以,

$$P_1 P_2 \cdots P_s A Q_1 Q_2 \cdots Q_t = \tilde{P}_1 \tilde{P}_2 \cdots \tilde{P}_{\tilde{s}} B \tilde{Q}_1 \tilde{Q}_2 \cdots \tilde{Q}_{\tilde{t}}.$$

于是,

$$\tilde{P}_{\tilde{s}}^{-1} \cdots \tilde{P}_1^{-1} P_1 P_2 \cdots P_s A Q_1 Q_2 \cdots Q_t \tilde{Q}_{\tilde{t}}^{-1} \cdots \tilde{Q}_1^{-1} = B.$$

即A 与B 等价.